Markov chains

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Markov chains (an example of stochastic process)

$X_0, X_1, \dots X_n$ (a step further beyond iid) [finite n states]

n r.v.s, not iid.

markov property: $p(\underbrace{X_{n+1}=j}_{\text{future}} \mid \underbrace{X_n=i}_{\text{present}}, \underbrace{X_{n-1}=i_{n-1} \dots X_0=i_0}_{\text{past}})$

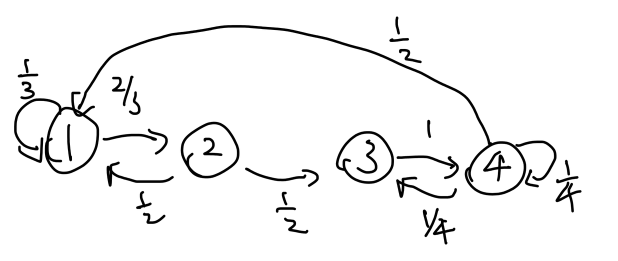
$= p(X_{n+1}=j \mid X_n=i) = q_{ij}$ (homogenous: doesn't depends on n/t) / first order markov chain

only go one step back

X_n. X_{n+1}

Markov assumption: future and past are conditionally indep given the present.

example:



v.z

transition matrix: probability of going **one** step future

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$M \times M$ M: # of states

don't need to draw the pic every time

application of Markov chain:

Markov chain Monte carlo: construct a Markov chain converge to a dist that you're interested in

LNN holds if iid

Markov chain one step further beyond iid

Suppose at time n, X_n has dist $\vec{s}_{1\times M}$ (M states) $[s_1, s_2 \dots s_i, \dots s_M]$

$\underbrace{p(X_{n+1}=j)}_{\text{marginal prob.}} = \sum_i \underbrace{p(X_{n+1}=j \mid X_n=i)}_{\to \text{ not iid.}} p(X_n=i)$ law of total prob

$= \sum_i s_i q_{ij}$ (sum of the j^{th} entry)

$M \times M$

$=$ $s_1 \cdot q_{1j} + s_2 q_{2j} + s_3 \cdot$

$\underbrace{[\cdot \ \underbrace{s_i}_{X_n=i} \ ]}_{\text{dist of } X_n} \begin{bmatrix} & s_i \ q_{\cdot j} & \end{bmatrix}_{M\times M} = [\ \underbrace{\sum s_i q_{ij}}_{X_{n+1}=j}\ ]$

$\vec{s}_{1\times M} \cdot Q_{M\times M} = \text{dist of } X_{n+1}$ ☆

- dist of X_{n+1}: $\vec{s} \cdot Q \to$ conditional prob $\xrightarrow{\text{law of total prob}}$ marginal dist $p(X_{n+1})$
- dist of X_{n+2}: dist of X_{n+1} · transition matrix $= \vec{s}\, Q \cdot Q = \vec{s} \cdot Q^2$

conditional prob.

transition matrix: multiple steps

∴ one step ahead: $p(X_{n+1}=j \mid X_n=i) = q_{ij}$

$p(A) = \sum_i p(A|B_i)p(B_i) \Rightarrow p(A|C) = \sum p(A|B_i,C)\,p(B_i|C)$

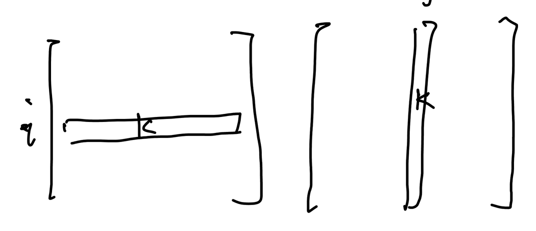
two step ahead: $p(X_{n+2}=j \mid X_n=i) = \sum_k \underbrace{p(X_{n+2}=j \mid X_{n+1}=k, X_n=i)}_{= p(X_{n+2}=j \mid X_{n+1}=k) \text{ Markov assumption}} \ p(X_{n+1}=k \mid X_n=i)\ \cancel{p(X_n=i)}$

↳ LHS is a conditional prob not marginal

$p(X_{n+2}=j) = \sum_k p(X_{n+2}=j \mid X_{n+1}=k)\ p(X_{n+1}=k)$

$\therefore p(X_{n+2}=j \mid X_n=i) = \sum_k \underbrace{p(X_{n+2}=j \mid X_{n+1}=k, X_n=i)}$ $= \sum_k q_{kj} \cdot \underbrace{q_{ik}}_{p(X_{n+1}=k \mid X_n=i)} \overset{\text{time order}}{=} \sum_k q_{ik} \cdot q_{kj} = \text{row } (i,j) \text{ of } Q^2$

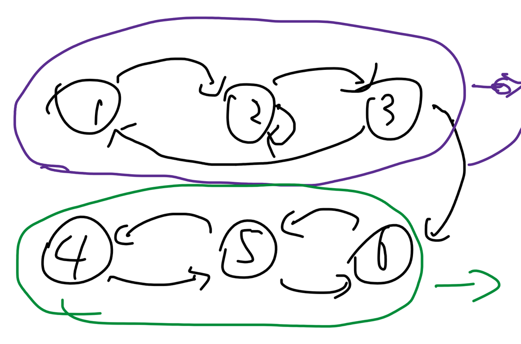
$= \sum_k p_{jk} \cdot p_{ik}$



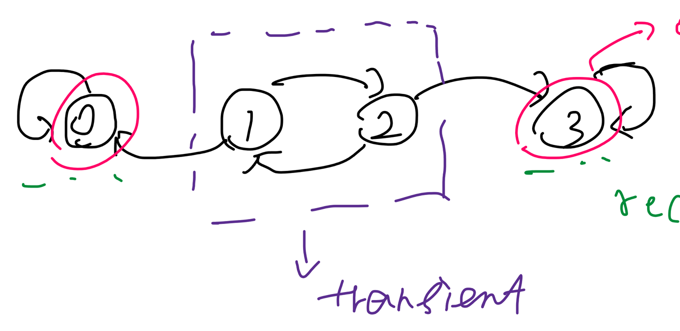
chain irreducible: if possible (with pos. prob.) to get from anywhere to anywhere.

state:
- recurrent, if starting there, chain has prob 1 of returning to that state. [can back infinite times]
- transient, otherwise [can come back, but eventually / infinite time, will stop]

case 1

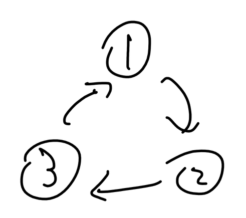


case 2



gambler's ruin.

case 3



periodic chain.

all reccurrent

stationary dist:

$\vec{s}$ pmf vector $1\times M$ is stationary for the chain if

$\vec{s}Q = \vec{s}$

Thm. For any irreducible Markov chain with finite states:

works for periodic chain
- (1) a stationary dist exist.
- (2) it's unique.
- (3) $s_i = \frac{1}{r_i}$ where r_i is the average return time (average steps) to return to i if ~~start~~ from state i
- (4) convergence

If Q^m is strictly pos. for some m,

↳ rule out periodic chain (each element in Q^m is pos.)

- for one element: $P(X_n = i) \longrightarrow s_i$ as $n \to \infty$. $[s_1, s_2 \dots s_i \dots s_M]$
- for a vector: $\underbrace{\vec{t}}_{\text{starting dist}} Q^n \longrightarrow \underbrace{\vec{s}}_{\text{stationary dist}}$

long-term behaviour.

How to find $\vec{s}$?

For reversible markov chains.

Defn. Markov chain with transition matrix $Q = q_{ij}$ is reversible.

if there is a prob. vector $\vec{s}$ s.t.

$s_i q_{ij} = s_j q_{ji}$ holds for all i, j.

if reversible w.r.t $\vec{s}$, then $\vec{s}$ is stationary.

example of reversible Markov chains: Random walk on an undirected network